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computers & industrial engineering
Computers & Industrial Engineering 47 (2004) 351–369
www.elsevier.com/locate/dsw

Testing process performance based on capability index C_{pk} with critical values☆

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Received 31 July 2001; accepted 16 March 2003
Available online 14 October 2004

Abstract

Process capability index C_{pk} has been widely used in the manufacturing industry as a process performance measure. In this paper, we investigate the natural estimator of the index C_{pk}, and show that under the assumption of normality its distribution can be expressed as a mixture of the chi-square and the normal distributions. We also implement the theory of hypothesis testing using the natural estimator of C_{pk}, and provide efficient *Maple* programs to calculate the p-values as well as the critical values for various values of α-risk, capability requirements, and sample sizes. The behavior of the p-values and critical values as functions of the distribution parameters are investigated to obtain tight critical values for reliable testing. Based on the test, we develop a simple and practical procedure for in-plant applications. The practitioners can use the proposed procedure to determine whether their process meets the preset capability requirement, and make reliable decisions.

Keywords: Process capability index; Testing hypothesis; Critical value

1. Introduction

Process capability indices, C_p, C_{pk} and C_{pm} have been proposed in the manufacturing industry and the service industry providing numerical measures on whether a process is capable of reproducing items within the specification limits preset in the factory. These indices have been defined as in the following,

☆ This manuscript was processed by Area Editor Elsayed A. Elsayed.

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0360-8352/$ - see front matter
doi:10.1016/j.cie.2003.03.001

where USL is the upper specification limit, LSL is the lower specification limit, μ is the process mean, σ is the process standard deviation, and T is the target value:

$$C_{\mathrm{p}} = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sigma},$$

$$C_{\mathrm{pk}} = \min\left\{\frac{\mathrm{USL} - \mu}{3\sigma}, \frac{\mu - \mathrm{LSL}}{3\sigma}\right\},$$

$$C_{\mathrm{pm}} = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sqrt{\sigma^2 + (\mu - T)^2}}.$$

In general, the process mean, μ, and the process standard deviation, σ, are unknown. But, in practice μ and σ can be estimated by the sample mean, $\bar{X} = \sum_{i=1}^{n} X_i/n$, and the sample standard $S = \{(n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\}^{1/2}$, to obtain the natural estimators of the three indices, $\hat{C}_{\mathrm{p}}$, $\hat{C}_{\mathrm{pk}}$ and $\hat{C}_{\mathrm{pm}}$. In order to calculate these estimators, however, sample data must be collected, and a great degree of uncertainty may be introduced into capability assessments owing to the sampling errors. The approach by simply looking at the calculated values of the estimated indices and then make a conclusion on whether the given process is capable, is unreliable as the sampling errors have been ignored. For normally distributed processes, Cheng (1994) has developed a testing procedure using the natural estimators $\hat{C}_{\mathrm{p}}$ and $\hat{C}_{\mathrm{pm}}$ for practitioners to use in determining if their process satisfies the targeted quality condition. However, no testing procedure for C_{pk} has ever been given. Numerous methods for constructing approximate confidence intervals of C_{pk} have been proposed in the literature. Examples include Chou, Owen, and Borrego (1990), Franklin and Wasserman (1991), Kushler and Hurley (1992), Nagata and Nagahata (1994), Tang, Than, and Ang (1997) and Zhang, Stenback, and Wardrop (1990) and many others.

The construction of exact confidence intervals and testing procedures for C_{pk} are complicated due to the fact that the distribution of $\hat{C}_{\mathrm{pk}}$ involves the joint distribution of two non-central t-distributed random variables (Kotz & Lovelace, 1998), or alternatively the joint distribution of folded-normal and chi-square random variables (Pearn, Kotz, & Johnson, 1992). No existing technique can provide exact confidence interval estimation and accurate capability testing up to date. In this paper, we first obtain an explicit form of the cumulative distribution function of the natural estimator $\hat{C}_{\mathrm{pk}}$, which can be expressed in terms of a mixture of the chi-square distribution and the normal distribution. We then implement the theory of hypothesis testing using the natural estimator of C_{pk}, and provide efficient *Maple* computer programs to calculate the p-values and the critical values. The behavior of the p-values and critical values as functions of the distribution parameters are investigated to obtain tight critical values for reliable testing. Based on the test we develop a simple step-by-step procedure for in-plant applications. The practitioners can use the proposed procedure to determine whether their process meets the preset capability requirement, and make reliable decisions.

2. Distribution of the estimated C_{pk}

Utilizing the identity $\min\{a,b\} = (a+b)/2 - |a-b|/2$, the index C_{pk} can be alternatively written as

$$C_{\mathrm{pk}} = \frac{d - |\mu - m|}{3\sigma}, \tag{1}$$

where $d=(\text{USL}-\text{LSL})/2$ is half of the length of the specification interval, $m=(\text{LSL}+\text{USL})/2$ is the mid-point between the lower and the upper specification limits. The natural estimator $\hat{C}_{\text{pk}}$ defined in the following

$$\hat{C}_{\text{pk}} = \frac{d - |\bar{X} - m|}{3S},$$

is obtained by replacing the process mean μ and the process standard deviation σ by their conventional estimators $\bar{X}$ and S, which may be obtained from a process that is demonstrably stable (under statistical control). Under the assumption of normality, Kotz and Johnson (1993) obtained the rth moment, and the first two moments as well as the mean and the variance of $\hat{C}_{\text{pk}}$. We now define $K=(n-1)S^2/\sigma^2$, $Z=n^{1/2}(\bar{X}-m)/\sigma$, $\xi=(\mu-m)/\sigma$, $Y=|Z|$. Then, the estimator $\hat{C}_{\text{pk}}$ can be rewritten as:

$$\hat{C}_{\text{pk}} = \frac{\sqrt{n-1}(3C_{\text{p}}\sqrt{n} - Y)}{3\sqrt{nK}}.$$

Under the assumption of normality, K is distributed as χ^2_{n-1}, a chi-square distribution with $n-1$ degrees of freedom. Further, since Z is distributed as the normal distribution $N(n^{1/2}\xi,1)$ with mean $n^{1/2}\xi$ and variance 1, then Y is distributed as the folded-normal distribution (see Leone, Nelson, & Nottingham, 1961). Noting that $C_{\text{p}}=d/(3\sigma)$, and $|\xi|=3(C_{\text{p}}-C_{\text{pk}})$. The probability density function of Y is

$$\begin{aligned} f_Y(y) &= \phi(y - \xi\sqrt{n}) + \phi(y + \xi\sqrt{n}) = \phi(y - |\xi|\sqrt{n}) + \phi(y + |\xi|\sqrt{n}) \\ &= \phi[y - 3(C_{\text{p}} - C_{\text{pk}})\sqrt{n}] + \phi[y + 3(C_{\text{p}} - C_{\text{pk}})\sqrt{n}], \quad y \geq 0, \end{aligned}$$

where $\phi(\cdot)$ is the probability density function of the standard normal distribution $N(0, 1)$. Using the method similar to that presented in Vännman (1997), we may obtain an exact form of the cumulative distribution function of $\hat{C}_{\text{pk}}$, under the assumption of normality. The cumulative distribution function of $\hat{C}_{\text{pk}}$ can be obtained and expressed in terms of a mixture of the chi-square distribution and the normal distribution. The cumulative distribution function of $\hat{C}_{\text{pk}}$ can be easily derived as follows. For $x>0$:

$$\begin{aligned} F_{\hat{C}_{\text{pk}}}(x) &= P(\hat{C}_{\text{pk}} \leq x) = P\left(\frac{\sqrt{n-1}(3C_{\text{p}}\sqrt{n} - Y)}{3\sqrt{nK}} \leq x\right) = 1 - P\left(\sqrt{nK} < \frac{\sqrt{n-1}(3C_{\text{p}}\sqrt{n} - Y)}{3x}\right) \\ &= 1 - \int_0^\infty P\left(\sqrt{nK} < \frac{\sqrt{n-1}(3C_{\text{p}}\sqrt{n} - Y)}{3x} | Y = y\right) f_Y(y)\mathrm{d}y \\ &= 1 - \int_0^\infty P\left(\sqrt{nK} < \frac{\sqrt{n-1}(3C_{\text{p}}\sqrt{n} - y)}{3x}\right) f_Y(y)\mathrm{d}y. \end{aligned}$$

Since K is distributed as χ^2_{n-1}, then

$$P\left(\sqrt{nK} < \frac{\sqrt{n-1}(3C_{\text{p}}\sqrt{n} - y)}{3x}\right) = 0,$$

for $y > 3C_p\sqrt{n}$ and $x > 0$. Hence:

$$F_{\hat{C}_{pk}}(x) = 1 - \int_0^{3C_p\sqrt{n}} P\left(\sqrt{n}K < \frac{\sqrt{n-1}(3C_p\sqrt{n} - y)}{3x}\right) f_Y(y)\mathrm{d}y$$

$$= 1 - \int_0^{3C_p\sqrt{n}} P\left(K < \frac{(n-1)(3C_p\sqrt{n} - y)^2}{9nx^2}\right) f_Y(y)\mathrm{d}y.$$

Therefore

$$F_{\hat{C}_{pk}}(x) = 1 - \int_0^{3C_p\sqrt{n}} G\left(\frac{(n-1)(3C_p\sqrt{n} - y)^2}{9nx^2}\right) f_Y(y)\mathrm{d}y, \tag{2}$$

for $x > 0$, where $f_Y(y) = \phi[y + 3(C_p - C_{pk})\sqrt{n}] + \phi[y - 3(C_p - C_{pk})\sqrt{n}]$, $G(\cdot)$ is the cumulative distribution function of the chi-square distribution χ^2_{n-1}.

3. Testing process capability

To test whether a given process is capable, we may consider the following statistical hypotheses testing:

H_0: $C_{pk} \le C$ (process is not capable),
H_1: $C_{pk} > C$ (process is capable).

We define the test $\phi^*(x)$, the decision making rule, as the following: $\phi^*(x) = 1$, if $\hat{C}_{pk} > c_0$; and $\phi^*(x) = 0$, otherwise. Thus, the test ϕ^* rejects the null hypothesis H_0 ($C_{pk} \le C$) if $\hat{C}_{pk} > c_0$, with type I error $\alpha(c_0) = \alpha$, the chance of incorrectly concluding an incapable process ($C_{pk} \le C$) as capable ($C_{pk} > C$). Given values of α and C, the critical value c_0 can be obtained by solving the equation $P(\hat{C}_{pk} \ge c_0 | C_{pk} = C) = \alpha$, using available numerical integration methods. Given a value of C (the capability requirement), the p-value corresponding to c^*, a specific value of $\hat{C}_{pk}$ calculated from the sample data, is (by Eq. (2)):

$$P(\hat{C}_{pk} \ge c^* | C_{pk} = C) = \int_0^{3C_p\sqrt{n}} G\left(\frac{(n-1)(3C_p\sqrt{n} - y)^2}{9n(c^*)^2}\right) \{\phi[y + 3(C_p - C)\sqrt{n}] + \phi[y - 3(C_p - C)\sqrt{n}]\}\mathrm{d}y. \tag{3}$$

Hence, given values of capability requirement C, the process characteristic parameter C_p, sample size n, and risk α, the critical value c_0 can be obtained by solving the following equation:

$$\int_0^{3C_p\sqrt{n}} G\left(\frac{(n-1)(3C_p\sqrt{n} - y)^2}{9nc_0^2}\right) \{\phi[y + 3(C_p - C)\sqrt{n}] + \phi[y - 3(C_p - C)\sqrt{n}]\}\mathrm{d}y = \alpha. \tag{4}$$

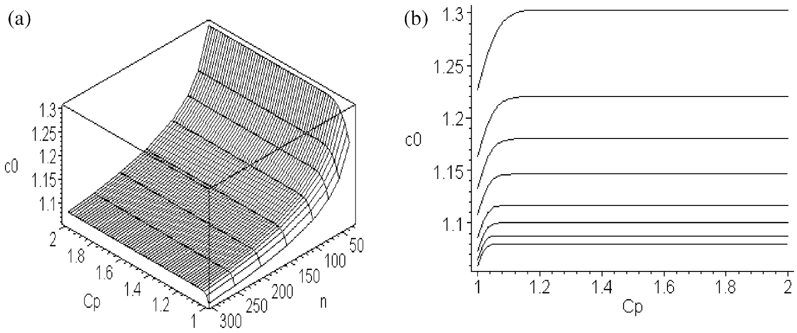

Fig. 1. (a) Surface plot of c_0 with $1.00 \le C_p \le 2.00$ and $30 \le n \le 300$ for $C_{pk} = 1.00$ and $\alpha = 0.05$. (b) Plots of c_0 versus C_p for $C_{pk} = 1.00$, $\alpha = 0.05$, and $n = 30$, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot).

3.1. Critical value c_0 and the parameter C_p

Since the process parameter σ is unknown, then the process distribution characteristic parameter, $C_p = d/(3\sigma)$ is also unknown, which has to be estimated in real application, naturally by substituting σ by he sample standard deviation S. Such approach introduces additional sampling errors from estimating C_p in finding the critical values, and certainly would make our approach (and all existing methods) less reliable. To eliminate the need for estimating the distribution characteristic parameter C_p, we examine the behavior of the critical values c_0 against the parameter C_p. We perform extensive calculations to obtain the critical values c_0 for $C_p = C(0.01)C''$, $C'' = (C+1)$, $n = 10(50)300$, $C = 1.00$, 1.33, 1.50, 1.67, 2.00, and $\alpha = 0.05$. Noting $C_p \ge C_{pk}$ and that parameter values we investigated, $C_p = C(0.01)C''$, cover a wide range of applications with process capability $C_{pk} \ge 0$. We find that the critical value c_0 (i) is increasing in C_p, and is decreasing in n, (ii) obtains its maximum at $C_p = C + 0.33$ in all cases, and (iii) stays the same for $C_p \ge C + 0.33$ for all C (with accuracy up to 10^{-6}). Further, we find that (iv) as the sample size $n \ge 30$ the critical value c_0 reaches its maximum at $C_p = C + 0.17$ and stays the same for $C_p \ge C + 0.17$ and as the sample size $n \ge 100$, $C_p = C + 0.12$ (with accuracy up to 10^{-4}). Hence, for practical purpose we may solve Eq. (4) with $C_p = C + 0.33$ to obtain the required critical values, without having to further estimate the parameter C_p. This approach ensures that the decisions made based on those critical values are more reliable than all existing methods. Efficient *Maple* programs are developed

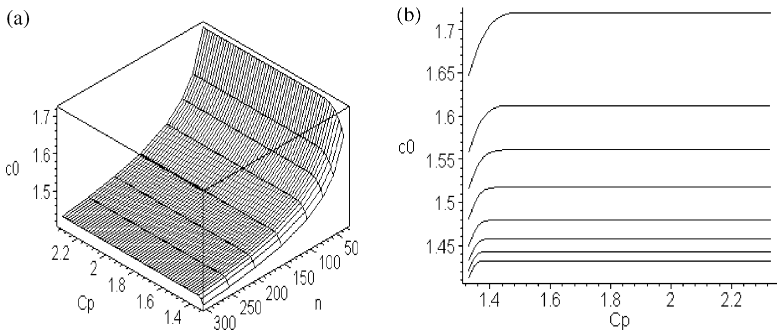

Fig. 2. (a) Surface plot of c_0 with $1.33 \le C_p \le 2.33$ and $30 \le n \le 300$ for $C_{pk} = 1.33$ and $\alpha = 0.05$. (b) Plots of c_0 versus C_p for $C_{pk} = 1.33$, $\alpha = 0.05$, and $n = 30$, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot).

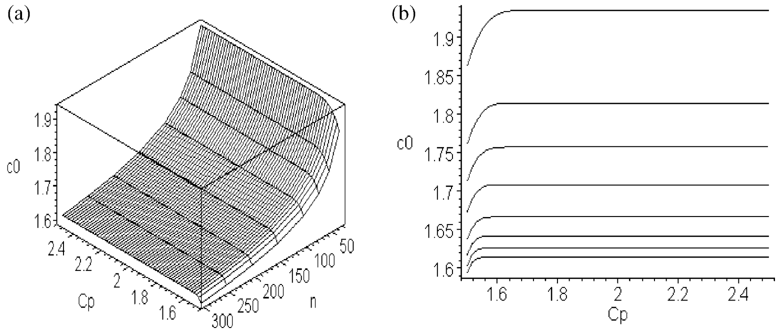

Fig. 3. (a) Surface plot of c_0 with $1.50 \le C_p \le 2.50$ and $30 \le n \le 300$ for $C_{pk} = 1.50$ and $\alpha = 0.05$. (b) Plots of c_0 versus C_p for $C_{pk} = 1.50$, $\alpha = 0.05$, and $n = 30, 50, 70, 100, 150, 200, 250, 300$ (top to bottom in plot).

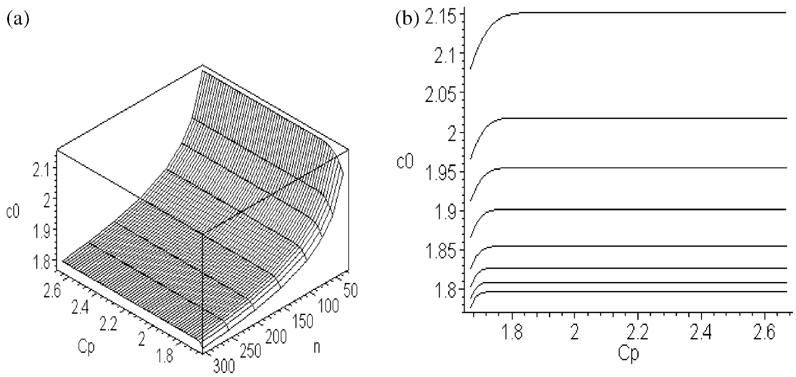

Fig. 4. (a) Surface plot of c_0 with $1.67 \le C_p \le 2.67$ and $30 \le n \le 300$ for $C_{pk} = 1.67$ and $\alpha = 0.05$. (b) Plots of c_0 versus C_p for $C_{pk} = 1.67$, $\alpha = 0.05$, and $n = 30, 50, 70, 100, 150, 200, 250, 300$ (top to bottom in plot).

for solving Eqs. (3) and (4) to obtain the p-value, and the critical value c_0. The two *Maple* programs are listed in the Appendices A and B.

Figs. 1(a)–5(a) display the surface plots of c_0 for $C \le C_p \le (C+1)$ and $30 \le n \le 300$ with $\alpha = 0.05$ for $C = 1.00$, 1.33, 1.50, 1.67, and 2.00, respectively. Figs. 1(b)–5(b) plot the curves of c_0 versus

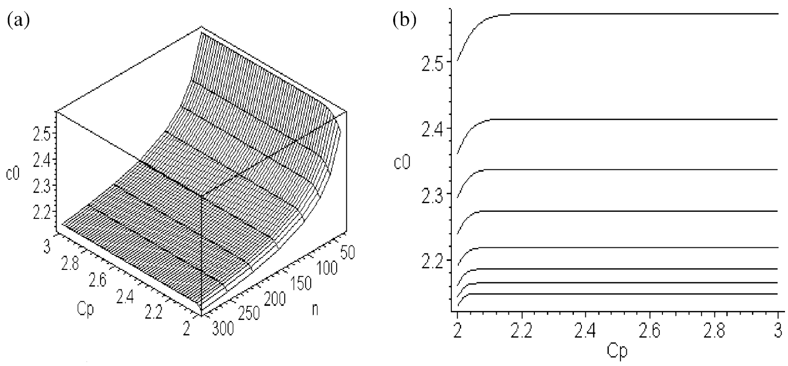

Fig. 5. (a) Surface plot of c_0 with $2.00 \le C_p \le 3.00$ and $30 \le n \le 300$ for $C_{pk} = 2.00$ and $\alpha = 0.05$. (b) Plots of c_0 versus C_p for $C_{pk} = 2.00$, $\alpha = 0.05$, and $n = 30, 50, 70, 100, 150, 200, 250, 300$ (top to bottom in plot).

Table 1
Critical values c_0 for $C=1.00$, $n=10(5)405$ and $\alpha=0.01$, 0.025, 0.05

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
10	2.141	1.877	1.686
15	1.748	1.618	1.493
20	1.621	1.469	1.399
25	1.525	1.423	1.342
30	1.461	1.399	1.303
35	1.415	1.337	1.275
40	1.379	1.309	1.252
45	1.351	1.286	1.235
50	1.328	1.268	1.220
55	1.308	1.253	1.208
60	1.292	1.240	1.197
65	1.278	1.228	1.188
70	1.266	1.219	1.180
75	1.255	1.210	1.173
80	1.245	1.202	1.167
85	1.236	1.195	1.161
90	1.228	1.188	1.156
95	1.221	1.183	1.151
100	1.214	1.177	1.147
105	1.208	1.172	1.143
110	1.203	1.168	1.139
115	1.198	1.164	1.135
120	1.193	1.160	1.132
125	1.188	1.156	1.129
130	1.184	1.153	1.126
135	1.180	1.149	1.124
140	1.176	1.146	1.121
145	1.173	1.143	1.119
150	1.170	1.141	1.117
155	1.166	1.138	1.115
160	1.163	1.136	1.113
165	1.161	1.133	1.111
170	1.158	1.131	1.109
175	1.155	1.129	1.107
180	1.153	1.127	1.106
185	1.151	1.125	1.104
190	1.148	1.123	1.103
195	1.146	1.122	1.101
200	1.144	1.120	1.100
205	1.142	1.118	1.098
210	1.140	1.117	1.097
215	1.139	1.115	1.096
220	1.137	1.114	1.095
225	1.135	1.112	1.094
230	1.133	1.111	1.092

(continued on next page)

Table 1 (*continued*)

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
235	1.132	1.110	1.091
240	1.130	1.109	1.090
245	1.129	1.107	1.089
250	1.127	1.106	1.088
255	1.126	1.105	1.087
260	1.125	1.104	1.087
265	1.123	1.103	1.086
270	1.122	1.102	1.085
275	1.121	1.101	1.084
280	1.120	1.100	1.083
285	1.119	1.099	1.082
290	1.117	1.098	1.082
295	1.116	1.097	1.081
300	1.115	1.096	1.080
305	1.114	1.095	1.079
310	1.113	1.095	1.079
315	1.112	1.094	1.078
320	1.111	1.093	1.077
325	1.110	1.092	1.077
330	1.109	1.091	1.076
335	1.109	1.091	1.076
340	1.108	1.090	1.075
345	1.107	1.089	1.074
350	1.106	1.089	1.074
355	1.105	1.088	1.073
360	1.104	1.087	1.073
365	1.104	1.087	1.072
370	1.103	1.086	1.072
375	1.102	1.085	1.071
380	1.101	1.085	1.071
385	1.101	1.084	1.070
390	1.100	1.084	1.070
395	1.099	1.083	1.069
400	1.099	1.082	1.069
405	1.098	1.082	1.068

the parameter C_p, ($C \le C_p \le (C+1)$) for sample size $n=30$, 50, 70, 100(50)300 from top to bottom in plots, and $C=1.00$, 1.33, 1.50, 1.67, 2.00, with $\alpha=0.05$.

4. The test procedure

Tables 1–5 display critical values c_0 for $C=1.00$, 1.33, 1.50, 1.67, and 2.00, with sample sizes $n=10(5)405$, and α-risk$=0.01$, 0.025, 0.05. To judge if a given process meets the capability requirement, we first determine the value of C, the capability requirement, and the α-risk. Checking the appropriate table from Tables 1–5, we may obtain the critical value c_0 based on given values of α-risk, C,

Table 2
Critical values c_0 for $C=1.33$, $n=10(5)405$ and $\alpha=0.01$, 0.025, 0.05

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
10	2.811	2.468	2.220
15	2.345	2.129	1.967
20	2.132	1.970	1.845
25	2.007	1.875	1.771
30	1.924	1.810	1.720
35	1.863	1.763	1.683
40	1.817	1.727	1.654
45	1.718	1.698	1.631
50	1.751	1.674	1.612
55	1.726	1.655	1.592
60	1.705	1.638	1.583
65	1.687	1.623	1.571
70	1.671	1.610	1.561
75	1.657	1.599	1.552
80	1.644	1.589	1.544
85	1.633	1.580	1.536
90	1.623	1.571	1.529
95	1.613	1.564	1.523
100	1.605	1.557	1.518
105	1.597	1.551	1.513
110	1.590	1.545	1.508
115	1.583	1.540	1.503
120	1.577	1.534	1.499
125	1.571	1.530	1.495
130	1.566	1.525	1.492
135	1.561	1.521	1.488
140	1.556	1.517	1.485
145	1.551	1.514	1.482
150	1.547	1.510	1.479
155	1.543	1.507	1.477
160	1.539	1.504	1.474
165	1.536	1.501	1.472
170	1.532	1.498	1.469
175	1.529	1.495	1.467
180	1.526	1.493	1.465
185	1.523	1.490	1.463
190	1.520	1.488	1.461
195	1.517	1.486	1.459
200	1.515	1.483	1.458
205	1.512	1.481	1.456
210	1.510	1.479	1.454
215	1.507	1.477	1.453
220	1.505	1.476	1.451
225	1.503	1.474	1.450
230	1.501	1.472	1.448

(*continued on next page*)

Table 2 (*continued*)

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
235	1.499	1.470	1.447
240	1.497	1.469	1.446
245	1.495	1.467	1.444
250	1.493	1.466	1.443
255	1.491	1.464	1.442
260	1.490	1.463	1.441
265	1.488	1.462	1.439
270	1.486	1.460	1.438
275	1.485	1.459	1.437
280	1.483	1.458	1.436
285	1.482	1.456	1.435
290	1.480	1.455	1.434
295	1.479	1.454	1.433
300	1.477	1.453	1.432
305	1.476	1.452	1.432
310	1.475	1.451	1.431
315	1.474	1.450	1.430
320	1.472	1.449	1.429
325	1.471	1.448	1.428
330	1.470	1.447	1.427
335	1.469	1.446	1.427
340	1.468	1.445	1.426
345	1.467	1.444	1.425
350	1.466	1.443	1.424
355	1.465	1.442	1.424
360	1.464	1.441	1.423
365	1.463	1.441	1.422
370	1.462	1.440	1.422
375	1.461	1.439	1.421
380	1.460	1.438	1.420
385	1.459	1.438	1.420
390	1.458	1.437	1.419
395	1.457	1.436	1.418
400	1.456	1.435	1.418
405	1.455	1.435	1.417

and the sample size n. If the estimated value $\hat{C}_{pk}$ is greater than the critical value c_0 ($\hat{C}_{pk} > c_0$), then we conclude that the process meets the capability requirement ($C_{pk} > C$). Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, we would believe that $C_{pk} \leq C$.

STEP 1: Decide the definition of 'capable' (set the value of C), and the α-risk (normally set to 0.01, 0.025, or 0.05), the chance of wrongly concluding an incapable process as capable.
STEP 2: Calculate the value of $\hat{C}_{pk}$ from the sample.
STEP 3: Check the appropriate table (or run the *Maple* program in Appendix B), and find the critical value c_0 based on C, α-risk and n.

Table 3
Critical values c_0 for $C=1.50$, $n=10(5)405$ and $\alpha=0.01$, 0.025, 0.05

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
10	3.159	2.774	2.496
15	2.635	2.394	2.213
20	2.397	2.215	2.076
25	2.257	2.109	1.993
30	2.163	2.036	1.936
35	2.096	1.984	1.894
40	2.044	1.943	1.862
45	2.004	1.911	1.836
50	1.970	1.884	1.815
55	1.942	1.862	1.798
60	1.919	1.843	1.782
65	1.898	1.827	1.769
70	1.880	1.813	1.758
75	1.865	1.800	1.747
80	1.851	1.789	1.738
85	1.838	1.779	1.730
90	1.827	1.769	1.722
95	1.816	1.761	1.716
100	1.807	1.753	1.709
105	1.798	1.746	1.704
110	1.790	1.740	1.698
115	1.782	1.734	1.693
120	1.776	1.728	1.689
125	1.769	1.723	1.685
130	1.763	1.718	1.680
135	1.757	1.713	1.677
140	1.752	1.709	1.673
145	1.747	1.705	1.670
150	1.742	1.701	1.667
155	1.738	1.697	1.664
160	1.733	1.694	1.661
165	1.729	1.690	1.658
170	1.726	1.687	1.655
175	1.722	1.684	1.653
180	1.718	1.681	1.651
185	1.715	1.679	1.648
190	1.712	1.676	1.646
195	1.709	1.674	1.644
200	1.706	1.671	1.642
205	1.703	1.669	1.640
210	1.700	1.667	1.638
215	1.698	1.664	1.637
220	1.695	1.662	1.635
225	1.693	1.660	1.633

(*continued on next page*)

Table 3 (*continued*)

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
230	1.690	1.658	1.632
235	1.688	1.657	1.630
240	1.686	1.655	1.629
245	1.684	1.653	1.627
250	1.682	1.651	1.626
255	1.680	1.650	1.625
260	1.678	1.648	1.623
265	1.676	1.647	1.622
270	1.674	1.645	1.621
275	1.672	1.644	1.620
280	1.671	1.642	1.618
285	1.669	1.641	1.617
290	1.667	1.640	1.616
295	1.666	1.638	1.615
300	1.664	1.637	1.614
305	1.663	1.636	1.613
310	1.661	1.635	1.612
315	1.660	1.634	1.611
320	1.659	1.632	1.610
325	1.657	1.631	1.609
330	1.656	1.630	1.608
335	1.655	1.629	1.608
340	1.654	1.628	1.607
345	1.652	1.627	1.606
350	1.651	1.626	1.605
355	1.650	1.625	1.604
360	1.649	1.624	1.604
365	1.648	1.623	1.603
370	1.647	1.622	1.602
375	1.646	1.622	1.601
380	1.645	1.621	1.601
385	1.644	1.620	1.600
390	1.643	1.619	1.599
395	1.642	1.618	1.599
400	1.641	1.617	1.598
405	1.640	1.617	1.597

STEP 4: Conclude that the process is capable ($C_{pk} > C$) if $\hat{C}_{pk}$ value is greater than the critical value c_0 ($\hat{C}_{pk} > c_0$). Otherwise, we do not have enough information to conclude that the process is capable.

5. Conclusions

Process capability index C_{pk} has been the most popular index used in the manufacturing industry as a process performance measure. In this paper, we investigated the natural estimator of the index C_{pk},

Table 4
Critical values c_0 for $C=1.67$, $n=10(5)405$ and $\alpha=0.01$, 0.025, 0.05

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
10	3.508	3.081	2.773
15	2.926	2.660	2.459
20	2.662	2.461	2.307
25	2.507	2.343	2.215
30	2.404	2.263	2.152
35	2.329	2.205	2.106
40	2.272	2.160	2.070
45	2.227	2.124	2.042
50	2.190	2.095	2.018
55	2.159	2.070	1.999
60	2.113	2.049	1.982
65	2.110	2.031	1.967
70	2.090	2.016	1.955
75	2.073	2.002	1.943
80	2.057	1.989	1.933
85	2.043	1.978	1.924
90	2.031	1.968	1.916
95	2.019	1.958	1.908
100	2.009	1.950	1.901
105	1.999	1.942	1.895
110	1.990	1.935	1.889
115	1.982	1.928	1.884
120	1.974	1.922	1.878
125	1.967	1.916	1.874
130	1.960	1.911	1.869
135	1.954	1.906	1.865
140	1.948	1.901	1.861
145	1.943	1.896	1.857
150	1.937	1.892	1.854
155	1.933	1.888	1.851
160	1.928	1.884	1.848
165	1.923	1.880	1.845
170	1.919	1.877	1.842
175	1.915	1.873	1.839
180	1.911	1.870	1.836
185	1.907	1.867	1.834
190	1.904	1.864	1.831
195	1.900	1.862	1.829
200	1.897	1.859	1.827
205	1.894	1.856	1.825
210	1.891	1.854	1.823
215	1.888	1.852	1.821
220	1.885	1.849	1.819
225	1.883	1.847	1.817
230	1.880	1.845	1.815
235	1.878	1.843	1.814

(*continued on next page*)

Table 4 (*continued*)

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
240	1.875	1.841	1.812
245	1.873	1.839	1.811
250	1.871	1.837	1.809
255	1.868	1.835	1.808
260	1.866	1.834	1.806
265	1.864	1.832	1.805
270	1.862	1.830	1.803
275	1.860	1.829	1.802
280	1.858	1.827	1.801
285	1.857	1.826	1.800
290	1.855	1.824	1.798
295	1.853	1.823	1.797
300	1.851	1.821	1.796
305	1.850	1.820	1.795
310	1.848	1.819	1.794
315	1.847	1.817	1.793
320	1.845	1.816	1.792
325	1.844	1.815	1.791
330	1.842	1.814	1.790
335	1.841	1.813	1.789
340	1.839	1.811	1.788
345	1.838	1.810	1.787
350	1.837	1.809	1.786
355	1.836	1.808	1.785
360	1.834	1.807	1.784
365	1.833	1.806	1.783
370	1.832	1.805	1.783
375	1.831	1.804	1.782
380	1.830	1.803	1.781
385	1.828	1.802	1.780
390	1.827	1.801	1.780
395	1.826	1.800	1.779
400	1.825	1.800	1.778
405	1.824	1.799	1.777

and showed that under the assumption of normality its distribution can be expressed as a mixture of the chi-square and the normal distributions. We implemented the theory of hypothesis testing using the natural estimator of C_{pk}, and provided efficient *Maple* programs to calculate the p-values and the critical values for various α-risks, capability requirements, and sample sizes. We investigated the behavior of the critical values and showed that it is a stable function of the process parameter C_{p}. Consequently, tight critical values can be found, which ensures that the type-I error is no greater than the preset risk α. Based on the test, we developed a simple but practical procedure for in-plant applications. The proposed approach ensures that the decisions made based on those critical values are more reliable than all existing methods. The practitioners can use the proposed procedure to determine whether their process meets the preset capability requirement, and make reliable decisions.

Table 5
Critical values c_0 for $C=2.00$, $n=10(5)405$ and $\alpha=0.01$, 0.025, 0.05

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
10	4.186	3.678	3.312
15	3.493	3.176	2.937
20	3.178	2.940	2.756
25	2.994	2.799	2.647
30	2.871	2.704	2.572
35	2.782	2.634	2.517
40	2.714	2.581	2.475
45	2.660	2.539	2.441
50	2.616	2.504	2.413
55	2.580	2.475	2.390
60	2.549	2.450	2.370
65	2.522	2.429	2.353
70	2.498	2.410	2.337
75	2.478	2.393	2.324
80	2.459	2.378	2.312
85	2.443	2.365	2.301
90	2.428	2.353	2.291
95	2.414	2.342	2.282
100	2.402	2.332	2.274
105	2.390	2.322	2.267
110	2.380	2.314	2.260
115	2.370	2.306	2.253
120	2.361	2.299	2.247
125	2.352	2.292	2.241
130	2.344	2.285	2.236
135	2.337	2.279	2.231
140	2.330	2.273	2.227
145	2.323	2.268	2.222
150	2.317	2.263	2.218
155	2.311	2.258	2.214
160	2.305	2.254	2.210
165	2.300	2.249	2.207
170	2.295	2.245	2.203
175	2.290	2.241	2.200
180	2.286	2.237	2.197
185	2.281	2.234	2.194
190	2.277	2.230	2.191
195	2.273	2.227	2.189
200	2.269	2.224	2.186
205	2.266	2.221	2.183
210	2.262	2.218	2.181
215	2.259	2.215	2.179
220	2.255	2.212	2.177
225	2.252	2.210	2.174
230	2.249	2.207	2.172
235	2.246	2.205	2.170

(*continued on next page*)

Table 5 (*continued*)

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
240	2.243	2.202	2.168
245	2.240	2.200	2.166
250	2.238	2.198	2.165
255	2.235	2.196	2.163
260	2.233	2.194	2.161
265	2.230	2.192	2.160
270	2.228	2.190	2.158
275	2.226	2.188	2.156
280	2.223	2.186	2.155
285	2.221	2.184	2.153
290	2.219	2.183	2.152
295	2.217	2.181	2.151
300	2.215	2.179	2.149
305	2.213	2.178	2.148
310	2.211	2.176	2.147
315	2.209	2.175	2.145
320	2.208	2.173	2.144
325	2.206	2.172	2.143
330	2.204	2.170	2.142
335	2.202	2.169	2.141
340	2.201	2.167	2.140
345	2.199	2.166	2.138
350	2.198	2.165	2.137
355	2.196	2.164	2.136
360	2.195	2.162	2.135
365	2.193	2.161	2.134
370	2.192	2.160	2.133
375	2.190	2.159	2.132
380	2.189	2.158	2.131
385	2.188	2.157	2.131
390	2.186	2.156	2.130
395	2.185	2.155	2.129
400	2.184	2.153	2.128
405	2.183	2.152	2.127

Acknowledgements

The authors would like to thank the anonymous referees for their helpful comments and suggestions, which improved the paper.

Appendix A. *Maple* program for the *p*-values

An efficient *Maple* computer program is developed to calculate Eq. (3), and obtain the *p*-value for arbitrary given c^*, a specific value of $\hat{C}_{\mathrm{pk}}$ calculated from the sample data. The program is listed below, with input parameters set to: the capability requirement $C=1.00$, sample size $n=100$, distribution characteristic $C_p = C + 0.12 = 1.12$, the estimated index $c^* = 1.15$. The program gives the p-value corresponding to the sample data $c^* = 1.15$ as 0.046.

```
> #input values of parameters C, n, c*(=c1).
  #set Cp=C+0.33 for n<100, Cp=C+0.12 for n>=100.
C:=1.00; n:=100; c1:=1.15;
Cp:=C+0.12;
G:=(c1,y)->stats[statevalf,cdf,chisquare[n-1]]
  ((n-1)*(3*Cp*n∧0.5-y)^2/(9*n*c1^2));
f:=y->(stats[statevalf,pdf,normald](y+3*(Cp-C)*n^0.5)
  +stats[statevalf,pdf,normald](y-3*(Cp-C)*n^0.5));
pV:=c1->int(G(c1,y)*f(y),y=0..(3*Cp*n^0.5));
p_Value:=evalf(pV(c1));
```

The output is:
$C := 1.00 \quad n := 100 \quad c1 := 1.15$
$Cp := 1.12$

$$G := (c1, y) \rightarrow \mathrm{stats}_{\mathrm{statevalf,\ cdf,\ chisquare}_{n-1}}\left(\frac{1}{9}\,\frac{(n-1)(3Cpn^{.5}-y)^2}{nc1^2}\right)$$

$$f := y \rightarrow \mathrm{stats}_{\mathrm{statevalf,\ pdf,\ normald}}(y + 3(Cp - C)n^{.5}) + \mathrm{stats}_{\mathrm{statevalf,\ pdf,\ normald}}(y - 3(Cp - C)n^{.5})$$

$$pV := c1 \rightarrow \int_0^{3Cpn^{.5}} G(c1, y) f(y) \mathrm{d}y$$

p_Value: = 0.04588919290.

Appendix B. *Maple* program for the critical values

An efficient *Maple* computer program for solving Eq. (4), and calculating the critical value c_0, is listed below, with input parameters set to: the capability requirement $C = 1.00$, sample size $n = 38$, distribution characteristic $C_p = C + 0.33 = 1.33$, and the risk $\alpha = 0.05$. The program finds the critical value corresponding to $\alpha = 0.05$ as 1.261.

```
> #input values of parameters C, n, α.
  #set Cp=C+0.33 for n<100, Cp=C+0.12 for n>=100.
C:=1.00; n:=38; α:=0.05;
Cp:=C+0.33;
G:=(c0,y)->stats[statevalf,cdf,chisquare[n-1]]
  ((n-1)*(3*Cp*n^0.5-y)^2/(9*n*c0^2));
f:=y->(stats[statevalf,pdf,normald](y+3*(Cp-C)*n^0.5)
  +stats[statevalf,pdf,normald](y-3*(Cp-C)*n^0.5));
```

```
pV:=c0->int(G(c0,y)*f(y),y=0..(3*Cp*n^0.5));
cv:=proc(r::numeric)
local c0;
c0:=r;
  if evalf(pV(c0))<=α and evalf(pV(c0-0.001))>α
    then c0
  elif evalf(pV(c0))<α and evalf(pV(c0−0.001))<α
    then c0:=c0−0.001: cv(c0)
  else c0:=c0+0.001: cv(c0)
  end if
end proc:
#input initial value r (we take r=1.260 in this example).
critical_value:=cv(1.260);
```

The output is:
$C := 1.00 \quad n := 38 \quad \alpha := 0.05$
$Cp := 1.33$

$$G := (c0, y) \to \mathrm{stats}_{\mathrm{statevalf,\ cdf,\ chisquare}_{n-1}}\left(\frac{1}{9}\,\frac{(n-1)(3Cpn^{.5}-y)^2}{nc0^2}\right)$$

$$f := y \to \mathrm{stats}_{\mathrm{statevalf,\ pdf,\ normald}}(y + 3(Cp - C)n^{.5}) + \mathrm{stats}_{\mathrm{statevalf,\ pdf,\ normald}}(y - 3(Cp - C)n^{.5})$$

$$pV := c0 \to \int_0^{3Cpn^{.5}} G(c0, y) f(y)\mathrm{d}y$$

critical_Value: = 1.261.

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